Pair-Free Domino Tilings in Two and Three Dimensions

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Abstract

We investigate domino tilings in two dimensions and determine geometric conditions that force tilings to contain certain substructures. In particular, for a special class of regions in the square lattice, we derive an inequality involving the number of flippable pairs and "windmills" contained in a tiling. Finally, we extend the result to the three-dimensional cubic lattice.

1. Introduction

Let $\mathbb{Z}^d$ be the d-dimensional integer lattice graph. Given a subset $R \subseteq \mathbb{Z}^d$, we define a *domino tiling* of R to be a partition of the subset into disjoint adjacent pairs; in other words, it is a perfect matching of the subgraph induced by R. In this note, we determine whether or not certain subsets of the lattice admit tilings which do not include specified substructures. If T is a domino tiling of a subset $R \subseteq \mathbb{Z}^d$ and D is a collection of dominoes, we say that T contains a *substructure isomorphic to* D if there is a graph isomorphism $\phi\colon \mathbb{Z}^d \to \mathbb{Z}^d$ such that $\phi(\delta) \in T$ for each $\delta \in D$. A basic form of this question to consider is that for a square grid and the "pair" configuration

$$D_{\text{pair}} = \Big\{\{(0,0),(1,0)\},\{(0,1),(1,1)\}\Big\}. \tag{1.1}$$

It is not difficult to see that every tiling of the grid must contain a pair. Indeed, if a pair-free tiling T exists, we may assume, without loss of generality, that it contains the domino labeled 1 in Figure 1.1. It follows that T must also contain the domino labeled 2, and from there the one labeled 3, and so on. Eventually, this process forces a pair to be formed after the $(2n-2)$th domino is placed, yielding a contradiction.

The theory of domino tilings has been well-studied (see [ST95]), and is occasionally described in terms of tiling subsets of $\mathbb{R}^2$ with sets congruent to a 1×2 closed rectangle while allowing boundaries to overlap. Notably, it was proven by Thurston in [Thu90] that every simply connected region $\Omega \subseteq \mathbb{R}^2$ is *flip-connected*, in the sense that any two tilings can be obtained from one another by repeatedly locating a pair (in this setting, two dominoes

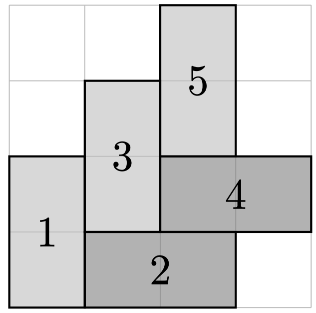

Figure 1.1: The impossibility of constructing of pair-free tiling of a 4×4 grid.

forming a 2×2 square) and performing a 90° rotation. In particular, if a tiling of a simply connected region Ω does not contain a pair, then it must be the *only* tiling that Ω admits. A general method for counting the number of domino tilings of a planar region due to P. W. Kasteleyn is given in [Kas61] and a generalization to bipartite graphs may be found in [LL93]. In principle, methods such as these may be used to show non-uniqueness of tilings, though we do not take this approach.

The main result of this paper resolves the pair-free tiling problem for a special class of subsets $R \subseteq \mathbb{Z}^2$ analogous to compact 2-manifolds in the plane. In Section 2, we develop basic notions and results for these subsets. In Section 3, we prove the main result via an argument reminiscent of the divergence theorem, equating the sum of a quantity over the region R with the sum of another quantity over the "boundary" of R. Finally, we apply the same type of argument in $\mathbb{Z}^3$ to obtain a similar result for grid graphs in Section 4.

1.1. Notation and conventions

As a slight abuse of language, we identify a subset $S \subseteq \mathbb{Z}^d$, with the subgraph of $\mathbb{Z}^d$ induced by S consisting of all edges between points in S. In this way, we may speak about graph-theoretic properties, such as degree or connectedness, for a general subset without confusion. Two subsets $S_1, S_2 \subseteq \mathbb{Z}^d$, are said to be *ambiently isomorphic*, which we denote by $S_1 \cong S_2$, if there exists a graph isomorphism $\phi\colon \mathbb{Z}^d \to \mathbb{Z}^d$ such that $\phi(S_1) = S_2$; this notion is extended to domino configurations in the same way.

2. Regular regions

Unless otherwise stated, our preferred choice of norm on $\mathbb{R}^d$ and $\mathbb{Z}^d$ will the maximum norm defined by $|x|_{\max} = \max |x_i|$. Given $p \in \mathbb{R}^d$, define its *box neighborhood* by

$$B(p) = \{q \in \mathbb{R}^d \mid |q - p|_{\max} < 1\}, \tag{2.1}$$

and given $p \in \mathbb{Z}^d$, define its *lattice neighborhood* by

$$B'(p) = \{q \in \mathbb{Z}^d \mid |q - p|_{\max} \leq 1\}. \tag{2.2}$$

For a subset $R \subseteq \mathbb{Z}^d$, define its *exterior* to be the complement $\mathrm{ext}(R) = \mathbb{Z}^d \setminus R$, its *interior* to be

$$\mathrm{int}(R) = \{p \in R \mid B'(p) \subseteq R\}, \tag{2.3}$$

and its *boundary* $\mathrm{bd}(R)$ to be the complement of the interior in R.

Definition 2.1. A subset $R \subseteq \mathbb{Z}^2$ is said to be a *regular region* if R is connected, $\mathrm{int}(R)$ is nonempty, and $\mathrm{bd}(R)$ is isomorphic to a disjoint union of cycle graphs.

Regular regions serve as an analogue to 2-manifolds with boundary in $\mathbb{R}^2$. If a regular region has only one boundary component, we say it is *simple.* The condition that the interior be nonempty guarantees that a regular region locally resembles a two-dimensional grid, in the sense given by the following proposition.

Proposition 2.2. *Let $R \subseteq \mathbb{Z}^2$ be a regular region. If e is an edge of R, then there exists a lattice square $Q \subseteq R$ containing e.*

Proof. Let us say an edge $e \subseteq R$ is *good* if there exists a square $Q \subseteq R$ containing e, and *bad* otherwise. Note that an edge contained in the lattice neighborhood of an interior point must already be good, so it suffices to show that edges in $\mathrm{bd}(R)$ are good.

We show that each boundary cycle contains at least one good edge. Suppose, for the sake of contradiction, that there exists a cycle $C \subseteq \mathrm{bd}(R)$ consisting of only bad edges. Each point in C must have exactly two of its lattice neighbors in C, and the remaining two in the exterior of R. It follows that C is a connected component of R, so $R = C$, which is a contradiction since R has nonempty interior. Hence, there must exist a good edge in any given boundary cycle.

Now, take a traversal $(q_i)_{i \in \mathbb{Z}/\ell\mathbb{Z}}$ of a boundary cycle C and define $e_i = \{q_i, q_{i+1}\}$ to be its ith edge. We conclude by proving the following claim:

($\star$) *If e_i is good, then e_{i+1} is also good.*

By choosing a suitable isometry, we may assume that $q_i = (1, 0)$ and $q_{i+1} = (0, 0)$. There are two cases:

1. Suppose $q_{i+2} = (-1, 0)$. By hypothesis, there exists a subsquare containing $(0, 0)$, which after a suitable reflection, contains $(0, 1)$. Since the boundary is a cycle, this cannot be another boundary point, so it must be an interior point.

   It follows that $e_{i+1} = \{q_{i+1}, q_{i+2}\} \subseteq B'(0, 1)$ is good.

2. Otherwise, after a suitable reflection, we may assume $q_{i+2} = (0, 1)$. By hypothesis, there exists a subsquare containing $(0, 0)$. If this square already contains q_{i+2}, then we are done.

   If not, then the square contains $(0, -1)$. Since the boundary is a cycle, this cannot be another boundary point, so it must be an interior point, which implies $(-1, 0) \in R$. For the same reason, this is also an interior point.

   It follows that $e_{i+1} = \{q_{i+1}, q_{i+2}\} \subseteq B'(-1, 0)$ is good.

Therefore, every edge in C is good, which is what we wanted to show. ■

Proposition 2.3. *Let $R \subseteq \mathbb{Z}^2$ be a regular region and C be a conneted component of* $\operatorname{bd}(R)$. *Then $|C| > 4$.*

Proof. Suppose, for the sake of contradiction, that $|C| \leq 4$. Since C is a cycle on the lattice graph, we must have $C \cong \{0, 1\}^2$. Note that C must have an adjacent point $q \in R \setminus C$. If this were not the case, then C would be a connected component of R, implying $R = C$, which is impossible since R must have nonempty interior. Since C is a connected component of the boundary, q must also be an interior point.

By choosing a suitable isometry, we may assume that $C = \{0, 1\}^2$ and $q = (-1, 0)$. Then $B'(-1, 0) \subseteq R$; of these points, $(-1, 1)$ and $(0, -1)$ are neighbors of C, so these must also be interior points. Then

$$B(0, 0) \subseteq B'(-1, 0) \cup B'(0, -1) \cup C \subseteq R, \tag{2.4}$$

which is a contradiction, since the origin is a boundary point. The conclusion follows. ■

2.1. Geometric realizations of subgraphs of $\mathbb{Z}^d$

It is useful to view $\mathbb{R}^d$ as a cell complex as follows: given $0 \leq k \leq d$ and a lattice k-cube $Q \cong \{0, 1\}^k \times \{0\}^{d-k}$, define the *cubical k-cell associated to Q* to be the relative interior of its convex hull; that is,

$$e_Q^k = \left\{ \sum_{q \in Q} \lambda_q q \;\middle|\; \lambda_q > 0, \sum_{q \in Q} \lambda_q = 1 \right\}. \tag{2.5}$$

These cells form a partition of $\mathbb{R}^d$ providing it with the structure of a cell complex. For a subset $S \subseteq \mathbb{Z}^d$, we may define its *geometric realization* $\|S\| \subseteq \mathbb{R}^d$ to be the union of all cells contained within S. Specifically, let $\mathcal{Q}_k(S)$ be the set of k-subcubes contained in S, and define

$$\|S\| = \bigcup_{0 \leq k \leq d} \left(\bigcup_{Q \in \mathcal{Q}_k(S)} e_Q^k \right). \tag{2.6}$$

Note that the realization of a subset $S \subseteq \mathbb{Z}^d$ automatically contains S, so two subsets of $\mathbb{Z}^d$ are disjoint if their realizations are. Furthermore, the converse also holds, since the cells partition $\mathbb{R}^d$.

Proposition 2.4. *Let $S_1, S_2 \subseteq \mathbb{Z}^d$. If $S_1 \cap S_2 = \varnothing$, then $\|S_1\| \cap \|S_2\| = \varnothing$.*

Proof. If $\|S_1\| \cap \|S_2\| \neq \varnothing$, then they share a common point p and a unique cell e_Q^k which contains p. It follows that $Q \subseteq S_1 \cap S_2$, so $S_1 \cap S_2 \neq \varnothing$. ■

Proposition 2.5. *Let $S \subseteq \mathbb{Z}^d$. If the subgraph induced by S is connected, then $\|S\|$ is connected as a topological space.*

Proof. Let $p, p' \in \|S\|$. There exist unique cells e_Q^k and $e_{Q'}^{k'}$ containing p and p' respectively. Since both cells are connected, p (resp. p') belongs to the same connected component as Q (resp. Q'). It suffices to show that these lattice subcubes also belong to the same connected component.

Since S is connected as a graph, there exists a path of adjacent points $q_1, \dots, q_n$ in S starting at Q and ending at Q'. Then $e^1_{\{q_i, q_{i+1}\}} \subseteq \|S\|$ is a cell whose boundary contains q_i, q_{i+1}, which therefore lie in the same connected component. By induction, all points in the path lie in the same connected component, which proves the claim. ■

Proposition 2.6. *Let $R \subseteq \mathbb{Z}^2$ be a regular region. Then* $\mathrm{bd}(\|R\|) = \|\mathrm{bd}(R)\|$.

Proof. Let $p \in \mathrm{bd}(\|R\|)$. There exists a unique cell $e_Q^k \subseteq \|R\|$ containing p. If some point $q \in Q$ was an interior point of R, then

$$p \in e_Q^k \subseteq B(q) \subseteq \|R\|, \tag{2.7}$$

which contradicts the fact that p lies in the topological boundary of $\|R\|$. It follows that each point of Q must be a boundary point, so $p \in \|\mathrm{bd}(R)\|$.

Conversely, let $p \in \|\mathrm{bd}(R)\|$. Then there exists a k-subcube $Q \subseteq \mathrm{bd}(R)$ such that $p \in e_Q^k$. By Proposition 2.3, we have $k < 2$. There are two cases:

1. If $k = 0$, then by applying a suitable isometry we may assume that $Q = \{p\} = \{(0,0)\}$. Suppose, for the sake of contradiction, that p was in the topological interior of $\|R\|$. Then there exists some ϵ with $0 < \epsilon < 1$ such that $\{-\epsilon, +\epsilon\}^2 \in \|R\|$, which implies that the lattice neighborhood of the origin is contained in R. Hence $(0,0)$ is an interior point of R, which is a contradiction.

   Therefore, p must be lie in the topological boundary of $\|R\|$.

2. If $k = 1$, then by applying a suitable isometry we may assume that $Q = \{(0,0), (1,0)\}$ and write $p = (x, 0)$ for some x satisfying $0 < x < 1$. Suppose, for the sake of

contradiction, that p was in the topological interior of $\|R\|$. Then there exists some ϵ with $0 < \epsilon < 1$ such that $(x, \pm\epsilon) \in \|R\|$, which implies that

$$\{(0,1), (0,-1), (1,1), (1,-1)\} \subseteq R. \tag{2.8}$$

Since each boundary point of R has exactly two neighboring boundary points, at least one of $(0, \pm 1)$ must be an interior point. In either case, we must have $(-1, 0) \in R$. We have two subcases:

(a) If $(-1, 0) \in \text{int}(R)$, then $(-1, \pm 1) \in R$.

(b) If $(-1, 0) \notin \text{int}(R)$, then $(0, \pm 1)$ must both be interior points, so $(-1, \pm 1) \in R$.

In both cases, the lattice neighborhood of the origin in contained in R, so $(0, 0)$ is an interior point of R, which is a contradiction.

Therefore, p must be lie in the topological boundary of $\|R\|$.

It follows that $\|\text{bd}(R)\| = \text{bd}(\|R\|)$. ■

The previous propositions imply the following corollary relating the boundary of a regular region to the boundary of its geometric realization.

Corollary 2.7. *Let R be a regular region. Then the number of connected components of* $\text{bd}(R)$ *is equal to the number of connected components of* $\text{bd}(\|R\|)$.

2.2. The polygon associated to a regular region

We define a polygonal region of $\mathbb{R}^2$ as a space locally modeled on "wedges". In particular, let $\arg\colon \mathbb{R}^2 \to [0, 2\pi)$ be a branch of the argument function, and for $\theta \in (-\pi, \pi]$, define the *closed sector with exterior angle* θ to be the subset

$$W_\theta = \{z \mid 0 \leq \arg(z) \leq \pi + \theta\} \subseteq \mathbb{R}^2 \tag{2.9}$$

For a subset $S \subseteq \mathbb{R}^2$ and $p \in S$, define a *sector chart* at p to be a tuple (U, V, ϕ) consisting of an open neighborhood U of p, an open neighborhood V of the origin, and an isometry

$$\phi\colon U \cap S \to V \cap W_\theta \tag{2.10}$$

or some $\theta \in (-\pi, \pi]$ which takes p to the origin. If such a chart exists, then $\theta = \theta_S(p)$ is uniquely determined, and is said to be the *exterior angle* at p. With this, we define a *polygonal region* to be a subset $P \subseteq \mathbb{R}^2$ which admits a sector chart at each point $p \in P$. The exterior angle function partitions the region into three sets:

1. a set $\text{int}(P)$ of *interior points*, whose exterior angles are all equal to π,

2. a set $\mathrm{bd}(P)$ of *boundary points*, whose exterior angles are all equal to 0, and
3. a set $\mathrm{cr}(P)$ of *corner points* or *vertices*, whose exterior angles are not equal to π or 0.

Note that the interior and boundary of a polygonal region as defined above agree with the usual topological definitions. Within a sector chart, there must be at most one corner point, so $\mathrm{cr}(P)$ is isolated. Thus, for a bounded polygonal region, there exist finitely many corners. We reformulate a well-known special case of the Gauss-Bonnet theorem in terms of the exterior angle function.

Theorem 2.8. *Let $P \subseteq \mathbb{R}^2$ be a connected, bounded, polygonal region with b boundary components. Then*

$$\sum_{p \in \mathrm{cr}(P)} \theta_P(p) = 2(2-b)\pi. \tag{2.11}$$

Proof. See [Car76]. ∎

3. Pair-free tilings of regular regions

Before stating the main theorem, we first prove a lemma instrumental to our upcoming argument. To begin, define a *domino indicator function* as follows: given a domino $\delta \subseteq \mathbb{Z}^d$ and a cube $Q \cong \{0,1\}^d$, let

$$\iota(\delta; Q) = \begin{cases} +1 & \text{if } \delta \not\subseteq Q \\ -1 & \text{if } \delta \subseteq Q \end{cases}. \tag{3.1}$$

In other words, the indicator takes on the value $+1$ when a domino is "leaving" the cube, and -1 when the domino is fully inside the cube.

Lemma 3.1. *Let $R \subseteq \mathbb{Z}^2$ be a regular region. Then $\|R\|$ is a polygonal region. Furthermore, if $q \in \mathrm{bd}(R)$, then*

$$\sum_{\substack{Q \subseteq R \\ Q \ni q}} \iota(\delta; Q) \leq -\frac{2\theta_{\|R\|}(q)}{\pi}. \tag{3.2}$$

Proof. By Proposition 2.6, $\mathrm{bd}(\|R\|)$ is covered by the collection of box neighborhoods centered at each point of $\mathrm{bd}\,R$. Since a sector itself is a polygonal region, it suffices to show that for each $q \in \mathrm{bd}\,R$, there exists some $\theta(q) \in (-\pi, \pi]$ satisfying (3.2) such that

$$B(q) \cap \|R\| \cong B(0) \cap W_{\theta(q)}. \tag{3.3}$$

Let q' be a neighboring boundary point. By Proposition 2.2, there exists a subsquare $Q_1 \subseteq R$ containing both q and q', and by choosing a suitable isometry, we may assume that

$$q = (0,0), \quad q' = (1,0); \quad Q_1 = \{(0,0),(1,0),(0,1),(1,1)\}. \tag{3.4}$$

There are three cases:

1. Suppose $(0,1) \in \operatorname{bd}(R)$. We show that $(0,-1) \notin R$ and $(-1,0) \notin R$. Indeed, if one of these points belongs to R, then it must be an interior point, since the origin cannot have another neighboring boundary point. This implies that both points belong to R, since they are each contained in the other's lattice neighborhood. Then

$$B'(0,0) \subseteq Q_1 \cap B'(0,-1) \cup B'(-1,0) \subseteq R, \tag{3.5}$$

which implies the origin is an interior point. This is a contradiciton, so $(0,-1) \notin R$ and $(-1,0) \notin R$. It follows that $B(q) \cap \|R\| = B(0) \cap W_{\pi/2}$. Furthermore, there is exactly one subsquare containing the origin, namely

$$Q_1 = \{(0,0), (+1,0), (0,+1), (+1,+1)\}. \tag{3.6}$$

Any domino $\delta \subseteq R$ containing q must be contained within a subsquare by Proposition 2.2; the only such one is Q_1. Therefore,

$$\sum_{\substack{Q \subseteq R \\ Q \ni q}} \iota(\delta, Q) = -1, \tag{3.7}$$

so the inequality (3.2) is also satisfied.

2. Suppose $(-1,0) \in \operatorname{bd}(R)$. Then $(0,1)$ must be an interior point, since it cannot be another boundary point. It follows that $(-1,1) \in R$. Additionally, $(0,-1) \notin R$, for if it did belong to R, then it would also be an interior point, so

$$B'(0,0) \subseteq B'(0,1) \cup B'(0,-1) \subseteq R, \tag{3.8}$$

which implies the origin is an interior point. This is a contradiction, so $(0,-1) \notin R$. It follows that $B(q) \cap \|R\| = B(0) \cap W_{\pi}$. Furthermore, there are exactly two subsquares containing the origin, namely

$$Q_1 = \{(0,0), (+1,0), (0,+1), (+1,+1)\} \tag{3.9}$$
$$Q_2 = \{(0,0), (-1,0), (0,+1), (-1,+1)\}. \tag{3.10}$$

Any domino $\delta \subseteq R$ containing q must be contained within at least one of these subsquare by Proposition 2.2. Therefore,

$$\sum_{\substack{Q \subseteq R \\ Q \ni q}} \iota(\delta, Q) \leq -1 + 1 = 0, \tag{3.11}$$

so the inequality (3.2) is also satisfied.

3. Suppose $(0,-1) \in \mathrm{bd}(R)$. Then $(0,1)$ must be an interior point, since it cannot be another boundary point. It follows that $(-1,0) \in R$, which again must be an interior point, since it also cannot be another boundary point. Additionally, $(1,-1) \notin R$, for if it did belong to R, then

$$B'(0,0) \subseteq B'(0,1) \cup B'(-1,0) \cup \{(1,-1)\} \subseteq R, \tag{3.12}$$

which implies the origin is an interior point. This is a contradiction, so $(1,-1) \notin R$. It follows that $B(q) \cap \|R\| = B(0) \cap W_{-\pi/2}$. Furthermore, there are exactly three subsquares containing the origin, namely

$$Q_1 = \{(0,0),(+1,0),(0,+1),(+1,+1)\} \tag{3.13}$$
$$Q_2 = \{(0,0),(-1,0),(0,+1),(-1,+1)\} \tag{3.14}$$
$$Q_3 = \{(0,0),(-1,0),(0,-1),(-1,-1)\}. \tag{3.15}$$

Any domino $\delta \subseteq R$ containing q must be contained within at least one of these subsquares. Therefore,

$$\sum_{\substack{Q \subseteq R \\ Q \ni q}} \iota(\delta, Q) \leq -1+1+1 = 1, \tag{3.16}$$

so the inequality (3.2) is also satisfied.

The conclusion follows. ■

We now prove the main result. Recall that the *pair* configuration consists of two dominoes which share a long side as shown in Figure 3.1, which can be written as

$$D_{\text{pair}} = \Big\{\{(0,0),(1,0)\},\{(0,1),(1,1)\}\Big\}. \tag{3.17}$$

It turns out to be useful to consider another domino configuration, which we call a *windmill*, which consists of four dominoes sharing a corner such that no two share a long side, as shown in Figure 3.2. Formally, this can be written as

$$D_{\text{windmill}} = \Big\{\{(0,0),(1,0)\},\{(0,1),(0,2)\},\{(-1,1),(-2,1)\},\{(-1,0),(-1,-1)\}\Big\}. \tag{3.18}$$

We show that the topological/graph-theoretic structure of a regular region gives a constraint on the number of these substructures contained in a given tiling. In precise terms, for a tiling T, we define

$$\#\text{pairs}(T) = |\{S \subseteq T \mid S \cong D_{\text{pair}}\}| \tag{3.19}$$

and

$$\#\text{windmills}(T) = |\{S \subseteq T \mid S \cong D_{\text{windmill}}\}|. \tag{3.20}$$

as the number of substructures isomorphic to the pair or the windmill, respectively.

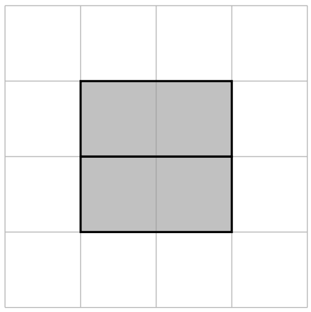

Figure 3.1: The pair configuration.

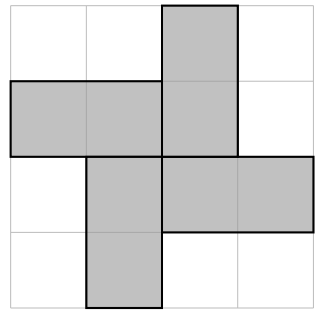

Figure 3.2: The windmill configuration.

Theorem 3.2. *Let $R \subseteq \mathbb{Z}^2$ be a regular region. If T is a domino tiling of R, then*

$$\#\text{windmills}(T) - \#\text{pairs}(T) \leq b(R) - 2, \tag{3.21}$$

where $b(R)$ is the number of boundary components of R.

Proof. Let $\delta_T \colon R \to T$ be the map which takes a point to the unique domino containing it. Then, given a subsquare $Q \cong \{0,1\}^2$, define the *divergence* of T at Q by

$$(\operatorname{div} T)(Q) = \frac{1}{4} \sum_{q \in Q} f(\delta_T(q); Q). \tag{3.22}$$

Let $\mathcal{P}$ and $\mathcal{W}$ be the set of pairs and windmills contained in T, respectively. If $Q \subseteq R$ is a subsquare, then we have three cases:

1. if Q is contained in an element of $\mathcal{P}$, then $(\operatorname{div} T)(Q) = -1$.
2. if Q is contained in an element of $\mathcal{W}$, then $(\operatorname{div} T)(Q) = +1$.
3. otherwise, there is at most one domino fully contained in Q, so

$$(\operatorname{div} T)(Q) = 1 - \tfrac{1}{2} \cdot |\{q \in Q \mid \delta_T(q) \subseteq Q\}| \geq 1 - \tfrac{1}{2} \cdot 1 \cdot 2 = 0. \tag{3.23}$$

Hence, the total divergence is bounded below by the number of excess windmills, that is,

$$\sum_{Q \subseteq R} (\operatorname{div} T)(Q) \geq \#\text{windmills}(T) - \#\text{pairs}(T). \tag{3.24}$$

For the rest of the argument, we calculate the sum of the divergence over each square in the region, reducing it to a sum of the "flux" of ι across the boundary.

Let $q \in R$. If q is an interior point, then there are exactly four subsquares $Q \cong \{0,1\}^2$ which contain it. Exactly two of these squares must fully contain $\delta_T(q)$, so

$$\frac{1}{4} \sum_{\substack{Q \subseteq R \\ Q \ni q}} \iota(\delta_T(q); Q) = \frac{1}{2} - \frac{1}{2} = 0. \tag{3.25}$$

Otherwise, if q is a boundary point, then by Lemma 3.1,

$$\frac{1}{4} \sum_{\substack{Q \subseteq R \\ Q \ni q}} \iota(\delta_T(q); Q) \leq -\frac{\theta_{\|R\|}(q)}{2\pi}. \tag{3.26}$$

Therefore, by Theorem 2.8 and Corollary 2.7, we have

$$\sum_{Q \subseteq R} (\operatorname{div} T)(Q) = \frac{1}{4} \sum_{q \in R} \sum_{Q \ni q} \iota(\delta_T(q); Q) \leq -\frac{1}{2\pi} \sum_{q \in \operatorname{cr}(\|R\|)} \theta_{\|R\|}(q) = b(R) - 2. \tag{3.27}$$

The conclusion follows. ■

Corollary 3.3. *If $R \subseteq \mathbb{Z}^2$ is simple, then R does not admit any pair-free domino tilings.*

3.1. Divergence of a domino tiling

In general, for a subset $R \subseteq \mathbb{Z}^d$ and a tiling T of R, we may define $\delta_T \colon R \to T$ to be the map taking a point to the unique domino containing it, and define the *divergence* of T by

$$(\operatorname{div} T)(Q) = \frac{1}{2^d} \sum_{q \in Q} \iota(\delta_T(q); Q). \tag{3.28}$$

The quantity's name is justified as follows. First, define $\sigma_T \colon R \to R$ to be the map taking a point to the other point in its containing domino. Then, if one realizes R as the union of closed cubes

$$\Omega_R = \bigcup_{q \in R} \overline{B_{1/2}(q)}, \tag{3.29}$$

and defines the vector field

$$X_T(p) = \begin{cases} \sigma_T(q) - q & \text{if } p \in B_{1/2}(q) \text{ for some } q \in R \\ 0 & \text{otherwise} \end{cases}, \tag{3.30}$$

then the divergence at a lattice cube $Q \cong \{0,1\}^d$ centered at $p \in \frac{1}{2}\mathbb{Z}^d$ is given by

$$(\operatorname{div} T)(Q) = \lim_{s \to 1^-} \frac{1}{\operatorname{vol}_{d-1}(\partial B_{2s}(p))} \int_{\partial B_{2s}(p)} X_T \cdot \mathbf{n} \tag{3.31}$$

This may be compared with divergence of a smooth vector field, in which case the limit is to be taken as $s \to 0$.

4. Domino tilings in three dimensions

We extend the previous tiling results for regular regions to three dimensional regions. For integers $n_1, n_2, n_3 \geq 2$, define

$$G(n_1, n_2, n_3) = \Big\{ (x_1, x_2, x_3) \in \mathbb{Z}^3 \;\Big|\; 0 \leq x_i \leq n_i \text{ for } i = 1, 2, 3 \Big\} \tag{4.1}$$

to be the *grid graph* with dimensions n_1, n_2, n_3. The grid graph can be divided into various regions based on how many coordinates are extremal (equal to 0 or n_i). Specifically, for a grid graph $G \subseteq \mathbb{Z}^3$, define

$$G^{[k]} = \Big\{ p \in G \mid \text{exactly } 3 - k \text{ coordinates of } p \text{ are extremal} \Big\}. \tag{4.2}$$

For $k = 0, 1, 2, 3$, these are called *corner*, *edge*, *face*, and *interior* points, respectively. In this setting, there are two domino configurations we wish to study: the familiar *pair*, consisting of two dominoes sharing a long face as shown in Figure 4.1, which we redefine here as

$$D_{\text{pair}} = \Big\{ \{(0,0,0), (1,0,0)\}, \{(0,1,0), (1,1,0)\} \Big\} \tag{4.3}$$

and a new configuration, the *loop*, consisting of three dominoes connected "head-to-tail" cyclically, as shown in Figure 4.2. Formally, this can be written as

$$D_{\text{loop}} = \Big\{ \{(0,0,0), (1,0,0)\}, \{(0,1,0), (0,1,1)\}, \{(1,0,1), (1,1,1)\} \Big\}. \tag{4.4}$$

Three-dimensional tilings with pairs and loops have been previously studied by C. Klivans and N. Saldanha in [KS25]. For the loop substructure, they define an operation called the *trit*, which modifies a tiling T by replacing a loop with a copy of its chiral twin.

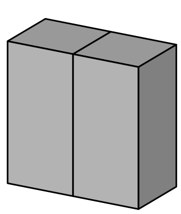

Figure 4.1: The pair configuration.

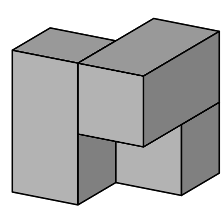

Figure 4.2: The loop configuration.

While it has been proven in [Thu90] that tilings of simply connected regions of the plane are flip-connected, it is not yet known whether or not even the tilings of $G(n_1, n_2, n_3)$ are connected by flips and trits. Since every grid graph has more than one domino tiling, a necessary condition for this to be true is for each tiling to contain at least one pair or loop. We show that this is the case.

Theorem 4.1. *Let $n_1, n_2, n_3 \geq 2$ be integers. If T is a domino tiling of $G(n_1, n_2, n_3)$, then T contains a pair or a loop.*

Proof. Assume, for the sake of contradiction, that there exists a tiling T which does not contain any substructure isomorphic to D_{pair} or D_{loop}. We first show that every subcube Q must contain at most two whole dominoes. By a suitable translation, we may assume that $Q = \{0,1\}^3$. Suppose there are already at least two dominoes fully contained within Q, and that they do not already form a pair. Then there are two cases:

1. If the two dominos are parallel, then by a suitable isometry we may take these to be

$$\{\delta_1, \delta_2\} = \Big\{\{(0,0,0),(0,0,1)\},\{(1,1,0),(1,1,1)\}\Big\}. \tag{4.5}$$

   There are two remaining candidates for a possible third domino $\delta_3 \subseteq Q$, but, as one can verify, both produce a substructure isomorphic to D_{pair}.

2. Otherwise, by a suitable isometry we may take these to be

$$\{\delta_1, \delta_2\} = \Big\{\{(0,0,0),(1,0,0)\},\{(0,1,0),(1,1,0)\}\Big\}. \tag{4.6}$$

   There are three remaining candidates for a possible third domino $\delta_3 \subseteq Q$, but, as one can verify, two produce a substructure isomorphic to D_{pair}, and the other yields a substructure isomorphic to D_{loop}.

It follows that

$$(\operatorname{div} T)(Q) = 1 - \tfrac{1}{4} \cdot \big|\{q \in Q \mid T(q) \subseteq Q\}\big| \geq 1 - \tfrac{1}{4} \cdot 2 \cdot 2 = 0. \tag{4.7}$$

for all subcubes Q. For the rest of the argument, we calculate the sum of $\operatorname{div} T$ over each cube in the region, reduce it to a sum over the boundary of the grid.

Let $q \in G^{[k]}$. If $k > 0$, then there exist 2^k subcubes containing q, and at least half of these cubes will fully contain $\delta_T(q)$. Hence,

$$\sum_{\substack{Q \subseteq G \\ Q \ni q}} \iota(\delta_T(q); Q) \leq 2^{k-1} - 2^{k-1} = 0. \tag{4.8}$$

Otherwise, if $k = 0$, then the one subcube containing q also fully contains $\delta_T(q)$, so

$$\sum_{\substack{Q \subseteq G \\ Q \ni q}} \iota(\delta_T(q); Q) = -1. \tag{4.9}$$

Overall, we obtain

$$\sum_{Q\subseteq G}(\operatorname{div} T)(Q) = \frac{1}{8}\cdot\sum_{q\in G}\sum_{Q\ni q} f(T(q),Q) \leq -1, \tag{4.10}$$

which contradicts (4.7). Therefore, every domino tiling T of the grid graph $G(n_1, n_2, n_3)$ must contain a substructure isomorphic to D_{pair} or D_{loop}. ■

References

[Car76] Manfredo Perdigão do Carmo. *Differential Geometry of Curves and Surfaces*. Prentice-Hall, 1976. ISBN: 0132125897.

[Kas61] P. W. Kasteleyn. "The statistics of dimers on a lattice: I. The number of dimer arrangements on a quadratic lattice". In: *Physica* 27.12 (1961), pp. 1209–1225. DOI: 10.1016/0031-8914(61)90063-5.

[KS25] Caroline J. Klivans and Nicolau C. Saldanha. "Domino tilings beyond 2D". In: *arXiv preprint arXiv:2507.22625* (2025). arXiv: 2507.22625 [math.CO].

[LL93] Elliott H. Lieb and Michael Loss. "Fluxes, Laplacians and Kasteleyn's theorem". In: *Duke Mathematical Journal* 71.2 (1993), pp. 337–363. DOI: 10.1215/S0012-7094-93-07114-1.

[ST95] Nicolau C. Saldanha and Carlos Tomei. "An overview of domino and lozenge tilings". In: *Resenhas IME-USP* 2.2 (1995), pp. 239–252.

[Thu90] William P. Thurston. "Conway's tiling groups". In: *The American Mathematical Monthly* 97.8 (1990), pp. 757–773. DOI: 10.2307/2324578.