On an Elementary Criterion for Monotonicity

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Abstract

In this note, we discuss a well-known Chern-Lashof-type criterion providing sufficient conditions for a domain to possess cross-sections homeomorphic to balls along at least one direction. We demonstrate how this can be used to prove that all polygons with five or fewer sides are monotone, answering in the affirmative a question posed by an anonymous user on a mathematics forum [Ano25].

1. Introduction

Let Ω be an open polygon in the plane and let $\theta \in \mathbb{S}^1$ be a direction vector. We say that Ω is *monotone* with respect to θ if its intersection with every line ℓ orthogonal to θ is either empty or an interval in ℓ. Intuitively, this means one can completely "hatch" the region with a pen without having to lift the pen for any hatch line, as shown in Figure 1. Monotonicity is a kind of generalization of convexity; if Ω is convex, then it is automatically monotone with respect to all directions. Typically, this notion is defined for polygons in terms of the number of crossings of a line with the boundary of the polygon, as done in [PS85]. The name is derived from the following fact (see [PS81]): if Γ is polygon monotone with respect to a direction θ, then the edges of Γ may be split into two continguous chains of vertices $v_1, \dots, v_n$ and $w_1, \dots, w_m$ such that $\langle v_i, \theta \rangle$ and $\langle w_j, \theta \rangle$ are monotone in i and j respectively. The definition of monotonicity may be extended to higher-dimensional domains (open sets) as follows. If $\Omega \subseteq \mathbb{R}^n$ is a bounded domain and $\theta \in \mathbb{S}^{n-1}$, we say that Ω is *monotone* with respect to θ if its intersection with every hyperplane H orthogonal to θ is either empty or a homeomorphic to the $(n-1)$-ball $\mathbb{B}^n$.

This note was prompted by an online discussion on shading polygons (see [Ano25]). While the referenced question can be resolved through purely elementary means, our goal is to demonstrate how it can be seen as a natural application a more general theorem relating curvature and monotonicity. This theorem is a well-known consequence of a formula in [CL57] given by S. S. Chern and R. K. Lashof, though we give a self-contained derivation for completeness.

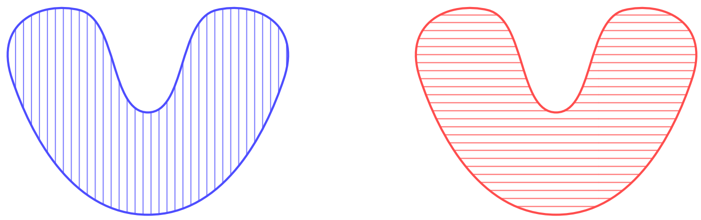

Figure 1: The V-shaped domain above is monotone horizontally (blue, vertical hatch lines), but not vertically (red, horizonal hatch lines).

1.1. Notation and conventions

We establish some useful notation, terminology, and conventions. Let $\mathbb{RP}^{n-1}$ be the quotient of $\mathbb{S}^{n-1}$ under the involution map $x \mapsto -x$ for all $x \in \mathbb{S}^{n-1}$. Since a domain $\Omega \subseteq \mathbb{R}^n$ is monotone with respect to a direction $\theta \in \mathbb{S}^{n-1}$ if and only if it is monotone with respect to $-\theta$, we may say that Ω is monotone with respect to the direction $[\theta] \in \mathbb{RP}^{n-1}$. The involution map on the sphere is an isometry, so the quotient carries an induced Riemannian metric. Denote the quotient map by $\rho\colon \mathbb{S}^{n-1} \to \mathbb{RP}^{n-1}$, which is a double cover and a local isometry.

For an smooth hypersurface $M \subseteq \mathbb{R}^n$ bounding a domain, the Gauss map is a surjective map $n\colon M \to \mathbb{S}^{n-1}$ which maps a point $p \in M$ to the (outward) normal vector at p. By composing with the double cover, we obtain a map $\nu\colon M \to \mathbb{RP}^{n-1}$ which we refer to as the *projectivized* Gauss map. The Gauss-Kronecker curvature of M is the unique real function K such that

$$n^*\omega_{\mathbb{S}^{n-1}} = K \cdot \omega_M, \tag{1.1}$$

where $\omega_{\mathbb{S}^{n-1}}$ and ω_M are the volume forms of $\mathbb{S}^{n-1}$ and M, respectively. The absolute Gauss-Kronecker curvature is given by Jacobian

$$|K(p)| = \sqrt{\det(\mathrm{d}n_p^* \circ \mathrm{d}n_p)}, \tag{1.2}$$

where $[\cdot]^*$ is the adjoint of a linear map between inner product spaces.

Finally, if Γ is a nondegenerate polygon, we use the convention that all exterior angles are given in the range $(-\pi, \pi)$.

2. Monotone domains

The main theorems of this paper rely on the following lemma which is analogous to [PS81, Theorem 1], which concerns polygons in the plane. Although the result we give fails to be a complete characterization of monotonicity in a given direction, it is enough for the purposes of later arguments.

Lemma 2.1. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with smooth boundary Γ, and let $\nu\colon \Gamma \to \mathbb{RP}^n$ be the projectivized Gauss map. If $\left|\nu^{-1}\{[\theta]\}\right| = 2$ for some $[\theta] \in \mathbb{RP}^{n-1}$, then Ω is monotone with respect to $[\theta]$.*

Proof. Consider the projection $\pi\colon \mathbb{R}^n \to \mathbb{R}$ defined by $\pi(x) = \langle x, \theta \rangle$ and let $h = \pi|_\Gamma$. At a critical point p of h, we have

$$\mathrm{d}h_p(v) = \langle v, \theta \rangle = 0$$

for all $v \in \mathrm{T}_p\Gamma$, which occurs if and only if $\nu(p) = [\theta]$. Since Γ is compact and not contained in a line, h attains a maximum and minimum at distinct points $p_+, p_- \in \Gamma$. By hypothesis, the preimage of $[\theta]$ under ν contains two elements, so there are no other critical values in $(\min h, \max h)$. By [Kam15, Theorem A], $h^{-1}\{t\}$ is homeomorphic to $\mathbb{S}^{n-2}$ whenever t is not an extreme value of h. Therefore, by the generalized Schoenflies theorem (see [Put25]), each slice $\Omega \cap \pi^{-1}\{t\}$ is homeomorphic to $\mathbb{B}^{n-1}$ whenever it is nonempty. The conclusion follows. ∎

We now prove the main theorems of the paper.

Theorem 2.2. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with smooth boundary Γ. If the total absolute Gauss-Kronecker curvature satisfies $\int_\Gamma |K| \,\mathrm{d}\Gamma < 2\operatorname{vol}(\mathbb{S}^{n-1})$, then Ω is monotone with respect to some direction.*

Proof. Let $n\colon \Gamma \to \mathbb{S}^{n-1}$ be the Gauss map and $\nu = \rho \circ n$, where $\rho\colon \mathbb{S}^{n-1} \to \mathbb{RP}^{n-1}$ is the projection map. Since ρ is a local isometry, the absolute Gauss-Kronecker curvature at a point $p \in \Gamma$ is given by the Jacobian

$$|K(p)| = \sqrt{\det(\mathrm{d}n_p^* \circ \mathrm{d}n_p)} = \sqrt{\det(\mathrm{d}\nu_p^* \circ \mathrm{d}\nu_p)} = |\mathrm{J}_p\nu|. \tag{2.1}$$

Define $\mu\colon \mathbb{RP}^{n-1} \to \mathbb{N}_{>0}$ by $\mu([\theta]) = |\nu^{-1}\{[\theta]\}|$. Then by (2.1) and the smooth coarea formula [Cha06], we have

$$\frac{1}{\operatorname{vol}(\mathbb{RP}^{n-1})} \int_{\mathbb{RP}^{n-1}} \mu \,\mathrm{d}\mathbb{RP}^{n-1} = \frac{1}{\frac{1}{2}\operatorname{vol}(\mathbb{S}^{n-1})} \int_\Gamma |K| \,\mathrm{d}\Gamma < 4, \tag{2.2}$$

so the average multiplicity of a direction $[\theta] \in \mathbb{RP}^{n-1}$ is strictly bounded above by 4. Note that $\deg_2(\nu) = 0$ since ν factors through a double cover. It follows that μ takes on positive even values almost everywhere, so such an average is attained only if $\mu([\theta]) = 2$ for some $\theta \in \mathbb{S}^{n-1}$. The conclusion follows from Lemma 2.1. ∎

By applying a standard smoothing argument, one may obtain an analogous result for polygons in the plane.

Theorem 2.3. *Let $\Omega \subseteq \mathbb{R}^2$ be a domain with polygonal boundary Γ. If the sum of the absolute values of the exterior angles of Γ is less than 4π, then Ω is monotone with respect to some direction.*

Proof. Let $\phi_1, \dots, \phi_n$ be the exterior angles of Γ. By "rounding" each of the corners of Ω, one may obtain a sequence $\Omega_i \to \Omega$ of domains with smooth boundaries Γ_i such that the multiplicities $\mu_i\colon \mathbb{RP}^1 \to \mathbb{N}_{>0}$ of the projectivized Gauss maps do not vary with i, and

$$\int_{\Gamma_i} |K| \,\mathrm{d}\Gamma_i = \sum_{k=1}^{n} |\phi_k|. \tag{2.3}$$

It follows from the proof of Theorem 2.2 that there exists a consistent direction $[\theta]$ for which each Ω_i is monotone. Then for a line ℓ normal to $[\theta]$, the slices $\Omega_i \cap \ell$ are intervals which converge to $\Omega \cap \ell$. Since the limit of a sequence of intervals is also an interval and $\Omega \cap \ell$ is open in ℓ, it must either be empty or homeomorphic to $\mathbb{B}^1$. The conclusion follows. ■

As a corollary, one may show that every polygon with five or fewer sides is monotone in at least one direction.

Corollary 2.4. *Let $\Omega \subseteq \mathbb{R}^2$ be a domain with n-sided polygonal boundary. If $n \leq 5$, then Ω is monotone with respect to some direction.*

Proof. Let $\theta_1, \ldots, \theta_n \in (-\pi, \pi)$ be the exterior angles of the boundary polygon. Suppose that j angles are nonnegative and k angles are negative. Since the sum of exterior angles is 2π, we have

$$\sum_{i=1}^{n} |\theta_i| = \sum_{\theta_i \geq 0} \theta_i - \sum_{\theta_i < 0} \theta_i < 2\pi \min(j-1, k+1) \leq 4\pi, \tag{2.4}$$

so Ω is monotone with respect to some direction. ■

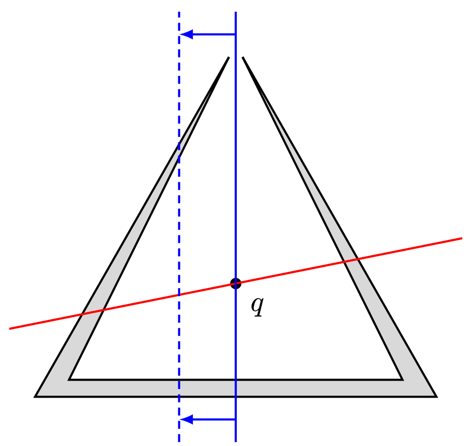

Figure 2: A hexagon which is not monotone with respect to any direction. Lines through the centroid q may be shifted to intersect the boundary in more than two points.

This result is sharp. For example, the hexagon in Figure 2 is not monotone with respect to any direction. Indeed, any line through q not passing through the "slit" must intersect the boundary in four points. On the other hand, any line through q passing through the slit may be translated left or right to achieve the same result.

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